trends, which therefore allows worst case conditions to be established, and also by showing quite accurately whether a given system will or will not be absolutely stable under these worst conditions. It should be remembered however, that the transfer function is based on small angles and perturbations. For any system where this is not the case, the quantitative merit of the transfer function should be reassessed by a comparison between its predictions and certain selected critical runs made on a full nonlinear computer simulation. Based on the results of the ITOS-1 spacecraft studies, various design changes were made to the control loop, pitch sensing equipment and inertia distribution, in order to optimize the stability boundaries under its various modes of operation. The in-orbit performance of the spacecraft now gives further verification of the method, and hence of its value as a design tool for dual-spin spacecraft.

References

[1] Landon, V. D. and Steward, B., "Nutational Stability of an Axisymmetric Body Containing a Rotor," *Journal of Spacecraft and Rockets*, Vol. 1, No. 6, June 1964, pp. 682–684.

[2] Likens, P. W., Tseng, G., and Mingori, D. L., "Stable Limit Cycles Due to Nonlinear Damping in Dual-Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 8, No. 6, June 1971, pp. 561–574.

[3] Likens, P. W., "Attitude Stability Criteria for Dual-Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 4, No. 2, Dec. 1967, pp. 1638–1643.

SEPTEMBER 1971 J. SPACECRAFT VOL. 8, NO. 9

Bearing Axis Wobble for a Dual Spin Vehicle

J. E. McIntyre* and M. J. Gianelli†
Hughes Aircraft Company, El Segundo, Calif.

Analytical expressions for the wobble motion of the bearing axis of a dual spin vehicle are derived in terms of the rotor static and dynamic unbalance. In the general case of arbitrary platform mass geometry, the bearing axis locus approximates a right cone with elliptical cross section. If the platform is configured so that the vehicle has equal transverse inertias in the platform fixed-coordinate system, the locus becomes circular and the wobble motion reduces to the familiar constant coning at spin speed which is encountered on single spin vehicles. Upper bounds on the mean and variance of the wobble angle are developed in terms of the statistical properties of the balancing process. The application of these bounds are indicated in an example problem which includes bearing run-out, ballast deployment uncertainty, propellant tank misalignment, bearing axis tilt and offset, and balancing machine static and dynamic residuals.

Nomenclature

$\mathbf{I}_1$ = rotor inertia matrix about rotor c.g.
$\mathbf{I}_2$ = platform inertia matrix about platform c.g.
Ω = relative rate of rotor with respect to platform
$\mathbf{r}_1$ = vector from system c.g. to rotor c.g.
$\mathbf{r}_2$ = vector from system c.g. to platform c.g.
m_1,m_2 = rotor and platform mass, respectively
$\mathbf{n}$ = unit vector along bearing axis
$\mathbf{r}$ = $\mathbf{r}_2 - \mathbf{r}_1$
m = reduced mass, $m_1m_2/(m_1 + m_2)$
$\boldsymbol{\epsilon}$ = rotor c.g. offset (normal to bearing axis)
$\epsilon_{x1},\epsilon_{y1}$ = components of $\boldsymbol{\epsilon}$ in rotor coordinate system
$\boldsymbol{\delta}$ = rotor principal axis misalignment with respect to bearing axis
δ_{x1},δ_{x2} = components of $\boldsymbol{\delta}$ in rotor coordinate system
z_0 = distance along bearing axis between platform and rotor c.g.
$\mathbf{h}$ = system angular momentum vector
$\boldsymbol{\omega}_1$ = angular rate of rotation of the rotor
$\boldsymbol{\omega}_2$ = angular rate of rotation of the platform
C_2 = symmetric platform axial inertia
$\mathbf{I}_3$ = $m(\mathbf{r}\cdot\mathbf{r} - \mathbf{r}\mathbf{r}^T)$
$\mathbf{J}$ = $\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 - C_2\mathbf{n}\mathbf{n}^T$
C_1 = rotor axial inertia
A_1 = rotor transverse inertia about rotor c.g.
A_2 = symmetric platform transverse inertia about platform c.g.
I_{x1z1} = $(C_1 - A_1)\delta_{y1} - m\epsilon_{x1}z_0$ } rotor cross products of inertia about system c.g.
I_{y1z1} = $(A_1 - C_1)\delta_{x1} - m\epsilon_{y1}z_0$ }
$\boldsymbol{\theta}$ = angular vector denoting orientation of $\mathbf{h}$ vector with respect to bearing axis
A = $\begin{cases} A_1 + A_2 + mz_0^2 \text{ in symmetric platform case} \\ A_1 + (I_{x2x2} + I_{y2y2})/2 \text{ in nonsymmetric case} \end{cases}$
ΔA = $(I_{x2x2} - I_{y2y2})/2$
$\mathbf{r}_0$ = vector from platform c.g. to the intersection of the bearing axis with the rotor c.g. plane
$\mathbf{I}_4$ = $\mathbf{I}_2 + m[\mathbf{r}_0\cdot\mathbf{r}_0 - \mathbf{r}_0\mathbf{r}_0^T] = \begin{pmatrix} I_{x2x2} & 0 & I_{x2z2} \\ 0 & I_{y2y2} & I_{y2z2} \\ I_{x2z2} & I_{y2y2} & I_{z2z2} \end{pmatrix}$
$\dot{\phi}$ = Ω
$c\phi,s\phi$ = $\cos\phi$, $\sin\phi$
$|A_0 - C_1|_{\rm MIN}$ = MINIMUM$\{|A + \Delta A - C_1|, |A - \Delta A - C_1|\}$
E = expectation operator
$\boldsymbol{\beta}$ = vector denoting angular tilt of bearing axis with respect to balancing machine spindle axis
β_x,β_y = transverse components of $\boldsymbol{\beta}$
$\boldsymbol{\alpha}$ = vector from spindle axis to bearing axis in the alignment plane of the balancing machine
l = distance from alignment plane to rotor c.g.

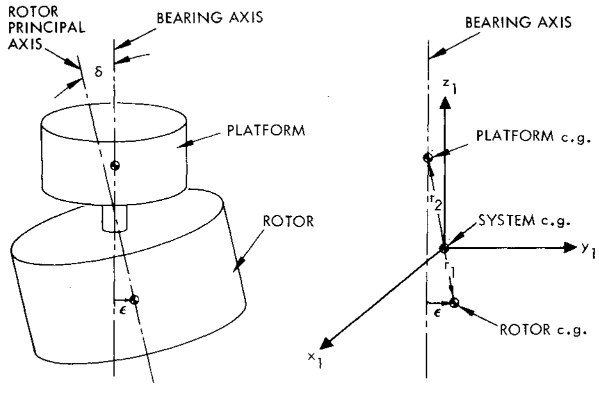
Received December 10, 1970; revision received May 18, 1971.
Index categories: Spacecraft Attitude Dynamics and Control; Spacecraft Configurational and Structural Design.

* Senior Staff Engineer, Guidance and Dynamics Department. Member AIAA.

† Member of the Technical Staff, Guidance and Dynamics Department.

Introduction

IN the past few years many papers and reports have appeared dealing with the nutational stability and control of dual spin vehicles.[1–5] A problem area of equal importance, as far as vehicle performances is concerned, is bearing axis



A. UNBALANCED ROTOR CONFIGURATION B. COORDINATE SYSTEM

Fig. 1 Vehicle geometry and coordinate system.

wobble. Despite the well-known nature of the problem, nothing yet on the subject seems to have made its way into the aeronautical literature. The present paper is offered to partially remedy this situation.

A dual spin vehicle consists of two near rigid bodies connected by means of a bearing which allows for a single degree of relative rotational freedom. One of the bodies, the rotor, is spun at a high rate to gyroscopically stabilize the vehicle. The second body, called the platform, is very nearly despun so that its payload instrument (antenna, camera, etc.) can be inertially pointed. Examples of such vehicles include the TACSAT, the INTELSAT IV and the family of Orbiting Solar Observatories.

During the course of the vehicle design, an attempt is made to locate the various rotor components so that the rotor is very nearly inertially symmetric about the bearing axis. Prior to launch, the rotor is carefully balanced so that its c.g. lies on the bearing axis and so that the bearing axis corresponds to a principal axis of inertia. In this configuration and with the platform despun, the vehicle motion will consist of an equilibrium spin about the bearing axis in which the spin vector, bearing axis and system angular momentum vector are coincident.

Because of errors in aligning the bearing and in balancing the rotor, the rotor c.g. will not lie on the bearing axis nor will the bearing axis correspond to a rotor principal axis. Hence, the equilibrium spin condition previously described will not result. Instead, the bearing axis will cone about the angular momentum vector. This coning motion is referred to as wobble, with the wobble angle θ defined as the angle which the bearing axis makes with the angular momentum vector.

The wobble motion is most easily visualized in terms of a single spinner. In this case, the vehicle is designed to spin about a certain axis (the geometric axis) by making it a maximum axis of inertia. However, because of balancing errors, the maximum inertia axis will deviate from the geometric axis by some small angle. Since the equilibrium motion consists of spin about the maximum axis with the spin and angular momentum vectors aligned, the geometric axis will cone at spin speed and at constant angle about the angular momentum vector.

For dual spin vehicles, the wobble motion of the bearing axis due to rotor unbalance is very similar to that experienced on single spin spacecraft; that is, most of the motion consists of a coning of the bearing axis about the $\mathbf{h}$ vector at rotor spin speed. However, depending on the mass properties of the platform, some variations in wobble amplitude and frequency can occur. In the following paragraphs an analytical expression for the wobble angle is developed as a function of rotor unbalance. Also, bounds on the wobble mean and variance are presented in terms of the statistical properties of the balancing process.

Exact Solution for an Axisymmetric Platform

Unlike the single spinner, the general motion of a dual spin vehicle is analytically intractable unless one of the bodies is inertially symmetric about the bearing axis. In this section it will be assumed that the platform satisfies this symmetry condition and an exact decription of the wobble motion will be given. In the following section, the symmetry assumption is relaxed and an approximate expression for the wobble angle is developed under the assumption that this angle is small.

In the developments which follow, it will be necessary to utilize two Cartesian coordinate systems; an $x_1y_1z_1$ system and an $x_2y_2z_2$ system. Both systems have their origin at the vehicle c.g. and both have their z axes parallel to the bearing axis. The $x_1y_1z_1$ system maintains fixed orientation with respect to the rotor while the $x_2y_2z_2$ system is fixed relative to the platform. In this section, only the rotor system will be used.

Consider the exaggerated dual spin configuration of Fig. 1 in which the platform is symmetric about the bearing axis, but in which the rotor principal axis and mass center are offset from the bearing axis. Letting $\boldsymbol{\omega}_1$ denote the angular rate of rotation of the rotor, the system angular momentum vector becomes

$$\mathbf{h} = \mathbf{I}_1\boldsymbol{\omega}_1 + \mathbf{I}_2(\boldsymbol{\omega}_1 - \Omega\mathbf{n}) + m\mathbf{r} \times \dot{\mathbf{r}} \tag{1}$$

Now the platform will be essentially despun which condition is represented mathematically by

$$\boldsymbol{\omega}_1 \cdot \mathbf{n} - \Omega = 0 \tag{2}$$

Also $\mathbf{n}$ is an eigenvector of I_2 (i.e., the bearing axis is a platform axis of symmetry). Hence,

$$\mathbf{I}_2\mathbf{n} = C_2\mathbf{n} \tag{3}$$

where C_2 is the platform axial inertia. In addition, the platform symmetry provides

$$\dot{\mathbf{r}} = \boldsymbol{\omega}_1 \times \mathbf{r}$$

from which it follows that

$$m\mathbf{r} \times \dot{\mathbf{r}} = m(\mathbf{r}\cdot\mathbf{r} - \mathbf{r}\mathbf{r}^T)\boldsymbol{\omega}_1 = \mathbf{I}_3\boldsymbol{\omega}_1 \tag{4}$$

Substituting Eqs. (2–4) into Eq. (1) yields the angular momentum vector as

$$\mathbf{h} = \mathbf{J}\boldsymbol{\omega}_1; \qquad \mathbf{J} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 - C_2\mathbf{n}\mathbf{n}^T \tag{5}$$

Under the assumption of zero external torque, the rate of change of $\mathbf{h}$ with respect to the $x_1y_1z_1$ rotor coordinate system satisfies

$$d\mathbf{h}/dt + \boldsymbol{\omega}_1 \times \mathbf{h} = 0 \Rightarrow \mathbf{J}d\boldsymbol{\omega}_1/dt + \boldsymbol{\omega}_1 \times \mathbf{J}\boldsymbol{\omega}_1 = 0 \tag{6}$$

But Eq. (6) is in precisely the same form as the equation for a single spinner and has the well-known equilibrium solution consisting of pure spin about a principal axis of $\mathbf{J}$. Thus, in equilibrium, the system spins about an axis which is both fixed in the rotor and fixed inertially (because of the constancy of $\mathbf{h}$). Because of the rotor c.g. offset and principal axis misalignments, $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ in Fig. 1, this axis will not correspond to the bearing axis. Rather, the bearing axis will cone about the spin axis at rotor spin frequency and at constant angle; that is, the wobble motion will exactly duplicate that occurring on a single spin spacecraft.

The evaluation of the wobble angle is easily accomplished. Since the spin axis is a principal axis of $\mathbf{J}$, it is only necessary to determine the coordinate system rotation which diagonalizes $\mathbf{J}$. The magnitude of this rotation is then the wobble amplitude. In what follows, an approximate expression for the matrix $\mathbf{J}$ will be developed under the assumption that the rotor offset $\boldsymbol{\varepsilon}$ and misalignment $\boldsymbol{\delta}$ are small. Hence, the resulting wobble expression will also be approximate and subject to the same smallness assumption on $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$. However, this assumption will be valid to a very high degree if any care at all is taken in the rotor balancing process.

To construct $\mathbf{J}$, note that the platform inertia can be written as

$$\mathbf{I}_2 = \begin{pmatrix} A_2 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & C_2 \end{pmatrix} \tag{7}$$

Hence,

$$\mathbf{I}_2 - C_2\mathbf{n}\mathbf{n}^T = \begin{pmatrix} A_2 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If the rotor inertia in the rotor principal axis system is given by

$$(\mathbf{I}_1)_{\text{prin}} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix} \tag{8}$$

and the principal axis is tilted with respect to the bearing axis by a small angular vector $\boldsymbol{\delta}$ with components

$$\boldsymbol{\delta} = (\delta_{x1},\delta_{y1},0)^T \tag{9}$$

then the inertia in the $x_1y_1z_1$ system becomes (to first order)

$$\mathbf{I}_1 = \begin{bmatrix} A_1 & 0 & (C_1 - A_1)\delta_{y1} \\ 0 & A_1 & -(C_1 - A_1)\delta_{x1} \\ (C_1 - A_1)\delta_{y1} & -(C_1 - A_1)\delta_{x1} & C_1 \end{bmatrix} \tag{10}$$

Finally, if the rotor c.g. offset $\boldsymbol{\epsilon}$ has components

$$\boldsymbol{\epsilon} = (\epsilon_{x1},\epsilon_{y},0)^T \tag{11}$$

the $\mathbf{r}$ vector from the platform to rotor c.g. becomes

$$\mathbf{r} = (\epsilon_{x1},\epsilon_{y1},z_0)^T \tag{12}$$

and the $\mathbf{I}_3$ matrix of Eq. (4) reduces to (first order)

$$\mathbf{I}_3 = m\begin{pmatrix} z_0^2 & 0 & -\epsilon_{x1}z_0 \\ 0 & z_0^2 & -\epsilon_{y1}z_0 \\ -\epsilon_{x1}z_0 & -\epsilon_{y1}z_0 & 0 \end{pmatrix} \tag{13}$$

Collecting all these results yields the inertia matrix $\mathbf{J}$ as

$$\mathbf{J} = \begin{pmatrix} A & 0 & I_{x1z1} \\ 0 & A & I_{y1z1} \\ I_{x1z1} & I_{y1z1} & C_1 \end{pmatrix} \tag{14}$$

where

$$A = A_1 + A_2 + mz_0^2 \tag{15a}$$

$$I_{x1z1} = (C_1 - A_1)\delta_{y1} - m\epsilon_{x1}z_0 \tag{15b}$$

$$I_{y1z1} = (A_1 - C_1)\delta_{x1} - m\epsilon_{y1}z_0 \tag{15c}$$

Now, to diagonalize $\mathbf{J}$, consider a small rotation vector $\boldsymbol{\theta}$ with components

$$\boldsymbol{\theta} = (\theta_{x1},\theta_{y1},0)^T$$

A simple calculation indicates that $\mathbf{J}$ will be diagonalized provided

$$\boldsymbol{\theta}_{x1} = I_{y1z1}/(A - C_1)$$

$$\boldsymbol{\theta}_{y1} = -I_{x1z1}/(A - C_1)$$

Hence, the wobble amplitude is given by

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{x1}^2 + \boldsymbol{\theta}_{y1}^2)^{1/2} = \{I_{x1z1}^2 + I_{y1z1}^2\}^{1/2}/|A - C_1| \tag{16}$$

where I_{x1z1} and I_{y1z1} are the products of inertia of the rotor about the system c.g.

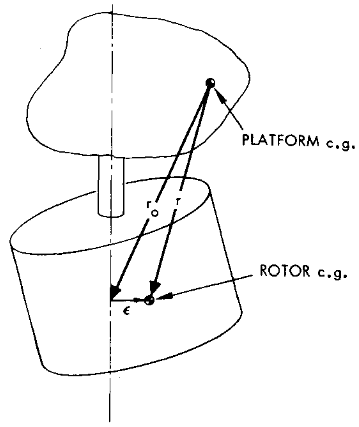


Fig. 2 Definition of the vector r_0.

Approximate Solution for Arbitrary Platform Mass Geometry

When the platform is axisymmetric, the wobble motion consists of a coning of the bearing axis about the spin axis at spin frequency and at constant amplitude. This description is exact; however, in developing the wobble amplitude in the preceding section, the approximation that $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ are small was introduced. In this section, the same assumption will be utilized to develop an approximate description of the wobble motion for a platform of arbitrary mass geometry.

Letting $\boldsymbol{\omega}_2$ denote the angular rate of rotation of the platform, the system angular momentum vector can be written as

$$\mathbf{h} = \mathbf{I}_1(\boldsymbol{\omega}_2 + \Omega\mathbf{n}) + \mathbf{I}_2\boldsymbol{\omega}_2 + m\mathbf{r} \times \dot{\mathbf{r}} \tag{17}$$

Now $\mathbf{r}$ is the vector from the platform to the rotor c.g. Let this vector be expressed as

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\epsilon}$$

where $\boldsymbol{\epsilon}$ is the rotor c.g. offset and where $\mathbf{r}_0$ is a vector from the platform c.g. to the intersection of the bearing axis with the horizontal plane containing the rotor c.g. The situation is pictured in Fig. 2. Thus,

$$m\mathbf{r} \times \dot{\mathbf{r}} = m\mathbf{r}_0 \times \dot{\mathbf{r}}_0 + m\mathbf{r}_0 \times \dot{\boldsymbol{\varepsilon}} + m\boldsymbol{\varepsilon} \times \dot{\mathbf{r}}_0 + m\boldsymbol{\varepsilon} \times \dot{\boldsymbol{\varepsilon}}$$

where

$$\dot{\mathbf{r}}_0 = \boldsymbol{\omega}_2 \times \mathbf{r}_0 \tag{18a}$$

$$\dot{\boldsymbol{\epsilon}} = (\boldsymbol{\omega}_2 + \Omega\mathbf{n}) \times \boldsymbol{\epsilon} \tag{18b}$$

Hence, Eq. (17) becomes

$$\mathbf{h} = \mathbf{I}_1(\boldsymbol{\omega}_2 + \Omega\mathbf{n}) + \mathbf{I}_4\boldsymbol{\omega}_2 + m(\mathbf{r}_0 + \boldsymbol{\epsilon}) \times [(\boldsymbol{\omega}_2 + \Omega\mathbf{n}) \times \boldsymbol{\epsilon}] + m\boldsymbol{\epsilon} \times (\boldsymbol{\omega}_2 \times \mathbf{r}_0) \tag{19}$$

where

$$\mathbf{I}_4 = \mathbf{I}_2 + m(\mathbf{r}_0\cdot\mathbf{r}_0 - \mathbf{r}_0\mathbf{r}_0)^T \tag{20}$$

If the rotor offset and misalignment were zero, the rotor would spin about the bearing axis with the platform having no angular motion. Hence, for small $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$, the platform rate $\boldsymbol{\omega}_2$ will also be small and a reasonable approximation is to neglect second and higher order effects in these variables. Thus, to first order in $\boldsymbol{\omega}_2$ and $\boldsymbol{\epsilon}$, Eq. (19) becomes

$$\mathbf{h} = \mathbf{I}_1(\boldsymbol{\omega}_2 + \Omega\mathbf{n}) + \mathbf{I}_4\boldsymbol{\omega}_2 + m\mathbf{r}_0 \times (\Omega\mathbf{n} \times \boldsymbol{\epsilon}) \tag{21}$$

This equation will next be resolved in the platform $x_2y_2z_2$ system.

The condition of a despun platform requires that the z component of $\boldsymbol{\omega}_2$ be zero. Therefore, $\boldsymbol{\omega}_2$ has the component representation

$$\boldsymbol{\omega}_2 = (\omega_{x2},\omega_{y2},0)^T \tag{22}$$

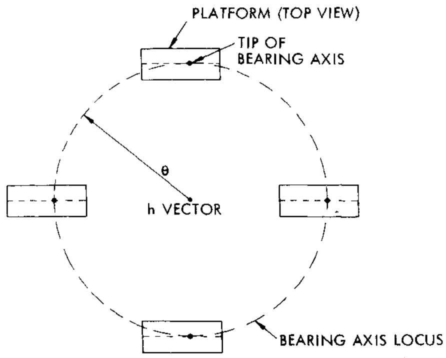


Fig. 3 Top view of wobble motion for $\Delta A = 0$.

Also, require that the x_2 and y_2 coordinate axes be chosen such that the inertia matrix $\mathbf{I}_4$ takes the form

$$\mathbf{I}_4 = \begin{pmatrix} I_{x_2x_2} & 0 & I_{x_2z_2} \\ 0 & I_{y_2y_2} & I_{y_2z_2} \\ I_{x_2z_2} & I_{y_2z_2} & I_{z_2z_2} \end{pmatrix} \tag{23}$$

The rotor inertia $\mathbf{I}_1$ has representation in the rotor coordinate system given by Eq. (10). Recognizing that the rotor and platform coordinate systems are related through the transformation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}; \qquad \dot{\phi} = \Omega \tag{24}$$

provides the $\mathbf{I}_1$ matrix in the platform coordinates as

$$\mathbf{I}_1 = \begin{pmatrix} A_1 & 0 & (C_1 - A_1)\delta_{y_1}c\phi - (A_1 - C_1)\delta_{x_1}s\phi \\ 0 & A_1 & (C_1 - A_1)\delta_{y_1}s\phi + (A_1 - C_1)\delta_{x_1}c\phi \\ (C_1 - A_1)\delta_{y_1}c\phi - (A_1 - C_1)\delta_{x_1}s\phi & (C_1 - A_1)\delta_{y_1}s\phi + (A_1 - C_1)\delta_{x_1}c\phi & C_1 \end{pmatrix} \tag{25}$$

Finally, letting $\mathbf{r}_0$ have components x_0, y_0, and z_0 and combining Eqs. (21–25) yields the $\mathbf{h}$ vector to first order as

$$\begin{pmatrix} h_{x2} \\ h_{y2} \\ h_{z2} \end{pmatrix} = \begin{Bmatrix} (A_1 + I_{x_2x_2})\omega_{x_2} + I_{x_1z_1}\Omega c\phi - I_{y_1z_1}\Omega s\phi \\ (A_1 + I_{y_2y_2})\omega_{y_2} + I_{x_1z_1}\Omega s\phi + I_{y_1z_1}\Omega c\phi \\ C_1\Omega + I_{x_2z_2}\omega_{x_2} + I_{y_2z_2}\omega_{y_2}\, m\Omega[y_0(s\phi\epsilon_{x_1} + c\phi\epsilon_{y_1}) + x_0(c\phi\epsilon_{x_1} - s\phi\epsilon_{y_1})] \end{Bmatrix} \tag{26}$$

where $I_{x_1z_1}$ and $I_{y_1z_1}$ are the rotor cross products of inertia about the system c.g. as defined in Eq. (15).

The dynamical equations

$$d\mathbf{h}/dt + \boldsymbol{\omega}_2 \times \mathbf{h} = 0$$

provide the first-order (steady-state) solution

$$\omega_{x_2} = \{(A + C_1 + \Delta A)/[C_1^2 - (A^2 - \Delta A^2)]\}\Omega(c\phi I_{x_1z_1} - s\phi I_{y_1z_1})$$
$$\omega_{y_2} = \{(A + C_1 - \Delta A)/[C_1^2 - (A^2 - \Delta A^2)]\}\Omega(s\phi I_{x_1z_1} + c\phi I_{y_1z_1}) \tag{27}$$

where the inertia A is the average vehicle transverse inertia

$$A = A_1 + (I_{x_2x_2} + I_{y_2y_2})/2 \tag{28}$$

and

ΔA is the average difference in the transverse inertias

$$\Delta A = (I_{x_2x_2} - I_{y_2y_2})/2 \tag{29}$$

Observe that the definition of A given in Eq. (28) reduces to that given in Eq. (15) when the platform is axisymmetric.

Let $\boldsymbol{\theta}$ be a small vector denoting the angular separation of the $\mathbf{h}$ vector and the z_2 (bearing) axis with component representation

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{x_2} \\ \theta_{y_2} \\ 0 \end{pmatrix} \tag{30}$$

Then to first order

$$\boldsymbol{\theta} = \begin{pmatrix} \omega_{x_2} \\ \omega_{y_2} \\ 0 \end{pmatrix} \tag{31}$$

and combining Eqs. (31) and (27) provides

$$\theta_{x_2} = \{(A + C_1 + \Delta A)/[C_1^2 - (A^2 - \Delta A^2)]\} \times (s\phi I_{x_1z_1} + c\phi I_{y_1z_1})$$
$$\theta_{y_2} = \{(A + C_1 - \Delta A)/(C_1^2 - [A^2 - \Delta A^2])\} \times (c\phi I_{x_1z_1} - s\phi I_{y_1z_1}) \tag{32}$$

In the special case in which ΔA is zero, the wobble amplitude becomes

$$\theta = (\theta_{x_2}^2 + \theta_{y_2}^2)^{1/2} = \{I_{x_1z_1}^2 + I_{y_1z_1}^2\}^{1/2}/|A - C_1| \tag{33}$$

That is, the wobble angle is constant in magnitude and, from the transformation in Eq. (24), constant in direction with respect to a rotor-fixed coordinate system. Hence, the bearing axis cones about the $\mathbf{h}$ vector at rotor spin speed. Although this result is the same as that given in Eq. (16) for the axisymmetric platform, its applicability is not restricted to the axisymmetric case. Rather, for ΔA to be zero and Eq. (33) to hold, the platform mass geometry can be arbitrary provided only that $I_{x_2x_2}$ and $I_{y_2y_2}$ as defined in Eq. (23) are equal. A top view of the resulting wobble motion is shown in Fig. 3.

For ΔA unequal to zero, the wobble angle sweeps out an ellipse in coning about the $\mathbf{h}$ vector with

$$\frac{\theta_{x_2}^2}{\left(\dfrac{C_1 + A + \Delta A}{C_1^2 - A^2 + \Delta A^2}\right)^2 \{I_{x_1z_1}^2 + I_{y_1z_1}^2\}} + \frac{\theta_{y_2}^2}{\left(\dfrac{C_1 + A - \Delta A}{C_1^2 - A^2 + \Delta A^2}\right)^2 \{I_{x_1z_1}^2 + I_{y_1z_1}^2\}} = 1 \tag{34}$$

Since ΔA is small in comparison with the quantity $C_1 + A$, the magnitude of θ can be approximated by

$$\theta = (\theta_{x_2}^2 + \theta_{y_2}^2)^{1/2} \cong \{I_{x_1z_1}^2 + I_{y_1z_1}^2\}^{1/2}/|A - C_1| \times \{1 - \Delta A/(A + C_1)\sin(2\phi - \gamma)\} \tag{35}$$
$$\gamma = \tan^{-1}\{(I_{x_1z_1}^2 - I_{y_1z_1}^2)/2I_{x_1z_1}I_{y_1z_1}\}$$

Note that the magnitude now contains a component which oscillates at twice the spin speed.

For the case in which $\Delta A \neq 0$, it is convenient to work with a circular bound on the amplitude of θ. From Eq. (34) it follows that

$$\theta \leq \theta_B = \{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}^{1/2}/|A_0 - C_1|_{\text{MIN}} \tag{36}$$

where $|A_0 - C_1|_{\text{MIN}}$ is given by

$$|A_0 - C_1|_{\text{MIN}} = \text{MINIMUM}\{|A + \Delta A - C_1|, |A - \Delta A - C_1|\} \tag{37}$$

that is, the bound is calculated using that transverse inertia (either $A + \Delta A$ or $A - \Delta A$) which makes the inertial difference a minimum. This bound, while not particularly tight, has the advantage of being similar in form to the exact expression in Eq. (33) for the $\Delta A = 0$ case.

As mentioned earlier, the wobble motion occurs because the rotor is imperfectly balanced about the bearing axis, which imbalance results in the rotor cross product terms $I_{x_1z_1}$ and $I_{y_1z_1}$ defined in Eq. (15). The usual procedure is to refer to that part of the cross product term resulting from the misalignment $\boldsymbol{\delta}$ as rotor dynamic unbalance, and that part resulting from the c.g. offset $\boldsymbol{\epsilon}$ as static unbalance. This separation of the contributing factors is a carry over from the single spinner case in which the wobble motion was a direct result of errors in the static and dynamic procedures used to balance the vehicle. As will be shown shortly, other factors in addition to balancing errors are responsible for the wobble motion of a dual spin vehicle.

Statistical Estimate of the Wobble Magnitude

To keep the wobble angle small, considerable care must be taken in balancing the rotor and in aligning the bearing axis with the rotor principal axis. Some of the methods used in aerospace balancing are surveyed in Ref. 6 while typical error sources and methods of suppressing them are discussed in Refs. 7 and 8, respectively. Of primary interest to the analysts are the expected value and variance of the wobble magnitude (or its bound). These moments are evaluated as follows.

Working with the bound expression of Eq. (36) provides

$$E\{\theta_B\} = (1/|A_0 - C_1|_{\text{MIN}})E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}^{1/2} \tag{38}$$

where E is the expectation operator. In the general case, this expected value will not have an analytical representation. However, an analytical upper bound is easily generated. From the Schwartz inequality it follows directly that

$$E\{|x|\} \leq (E\{x^2\})^{1/2}$$

and

$$E\{(|x| - E\{|x|\})^2\} \leq E\{x^2\}$$

Applying the previous results to Eq. (38) provide upper bounds on the first and second central moments of the form

$$E\{\theta_B\} \leq (1/|A_0 - C_1|_{\text{MIN}})[E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}]^{1/2} \tag{39a}$$

$$E\{(\theta_B - E\{\theta_B\})^2\} \leq (1/|A_0 - C_1|_{\text{MIN}})^2[E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}] \tag{39b}$$

In the special case in which $I_{x_1z_1}$ and $I_{y_1z_1}$ are zero mean independent gaussian variables with equal variance such that

$$E\{I_{x_1z_1}\} = E\{I_{y_1z_1}\} = E\{I_{x_1z_1}I_{y_1z_1}\} = 0$$
$$E\{I_{x_1z_1}{}^2\} = E\{I_{y_1z_1}{}^2\} = \sigma^2 \tag{40}$$

θ_B has a Rayleigh distribution with

$$E\{\theta_B\} = (1/|A_0 - C_1|_{\text{MIN}})(\pi/2)^{1/2}\sigma \tag{41}$$

$$E\{(\theta_B - E\{\theta_B\})^2\} = (1/|A_0 - C_1|_{\text{MIN}})\{2 - \pi/2\}\sigma^2 \tag{42}$$

Note that the upper bound on the mean in Eq. (39a) differs by approximately 25% from the exact value in Eq. (41) while the upper bound on the variance is too large by a factor of five.

In many applications the assumption that the cross products of inertia, $I_{x_1z_1}$ and $I_{y_1z_1}$ are gaussian and satisfy Eqs. (40) is a valid one; and in such cases, the wobble angle can be estimated using Eqs. (41) and (42). On the other hand, the bounding expressions of Eqs. (39) have a built-in safety factor in that they provide a conservative estimate regardless of the statistical distributions of $I_{x_1z_1}$ and $I_{y_1z_1}$. For this reason, Eqs. (39) will be used in the example problem calculations.

One further point regarding nomenclature is in order. The cross products $I_{x_1z_1}$ and $I_{y_1z_1}$ are due to a number of independent error sources. For example, the expected value

$$E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}$$

can usually be written

$$E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\} = \sum_{i=1}^{N} E\{[I_{x_1z_1}{}^{(i)}]^2 + [I_{y_1z_1}{}^{(i)}]^2\} \tag{43}$$

where the superscript i denotes the contribution from the ith error source. Defining the variables $\theta^{(i)}$ by means of the equation

$$\theta^{(i)} = (1/|A_0 - C_1|_{\text{MIN}})(E\{[I_{x_1z_1}{}^{(i)}]^2 + [I_{y_1z_1}{}^{(i)}]^2\})^{1/2} \tag{44}$$

the statistical bounds in Eqs. (39) become

$$E\{\theta_B\} \leq \left(\sum_{i=1}^{N} [\theta^{(i)}]^2\right)^{1/2} \tag{45a}$$

$$E\{(\theta_B - E\{\theta_B\})^2\} \leq \sum_{i=1}^{N} [\theta^{(i)}]^2 \tag{45b}$$

These expressions will prove useful in the example which follows.

Example Problem

To illustrate the methods of the preceding section, consider a symmetrical dual spin vehicle and with mass properties given in Table 1. Because of balancing errors and because of configuration differences between the balanced and in-orbit vehicles, the actual mass properties will deviate slightly from those in the table and the bearing axis will wobble about the $\mathbf{h}$ vector. The effects of these deviations on wobble amplitude are as follows:

Balance Machine Residual Errors

For illustrative purposes, assume a spinning table is used to null out the rotor cross products and to position the rotor c.g. on the table spindle axis. Because of a variety of errors, these quantities will have some nonzero residual value following the balancing process. Using superscript 1 to denote this effect yields

$$I_{x_1z_1}{}^{(1)} = (I_{x_1z_1})_{\substack{\text{RES}\\ \text{UNB}}} - mz_0\,(\epsilon_{x_1})_{\substack{\text{RES}\\ \text{OFFSET}}}$$

$$I_{y_1z_1}{}^{(1)} = (I_{y_1z_1})_{\substack{\text{RES}\\ \text{UNB}}} - mz_0\,(\epsilon_{y_1})_{\substack{\text{RES}\\ \text{OFFSET}}}$$

and assuming the offset and unbalance are zero mean independent random variables provides

$$[\theta^{(1)}]^2\cdot(A_0 - C_1)^2{}_{\text{MIN}} = E\{[I_{x_1z_1}{}^{(1)}]^2 + [I_{y_1z_1}{}^{(1)}]^2\} = E\{I_{x_1z_1}{}^2 + I_{y_1z_1}{}^2\}_{\substack{\text{RES}\\ \text{UNB}}} + (mz_0)^2E\{\epsilon_{x_1}{}^2 + \epsilon_{y_1}{}^2\}_{\substack{\text{RES}\\ \text{OFFSET}}} \tag{46}$$

Bearing Axis Tilt and Offset

The bearing axis will be slightly misaligned with respect to the spindle axis about which the rotor is balanced. Let this misalignment be represented by an angular tilt vector $\boldsymbol{\beta}$ and a position offset vector $\boldsymbol{\alpha}$ in the alignment plane as indicated

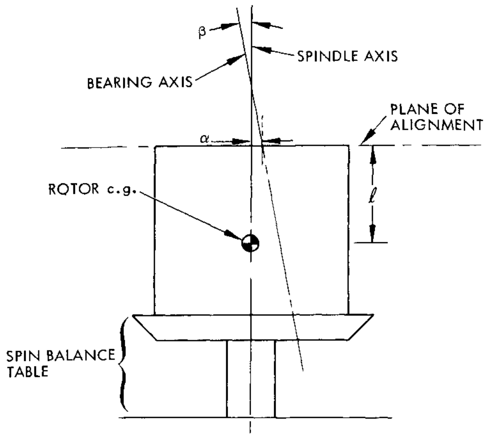


Fig. 4 Geometry of bearing axis tilt and offset.

in Fig. 4. With $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ having components

$$\boldsymbol{\beta} = \begin{pmatrix}\beta_x\\ \beta_y\\ 0\end{pmatrix}; \qquad \boldsymbol{\alpha} = \begin{pmatrix}\alpha_x\\ \alpha_y\\ 0\end{pmatrix}$$

it follows that

$$I_{x_1z_1}{}^{(2)} = -(C_1 - A_1)\beta_y + mz_0(\alpha_x - l\beta_y)$$
$$I_{y_1z_1}{}^{(2)} = -(A_1 - C_1)\beta_x + mz_0(\alpha_y + l\beta_x)$$

where l is the distance from the rotor c.g. to the alignment plane. If an addition $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are zero mean independent random variables, then

$$[\theta^{(2)}]^2 \cdot (A_0 - C_1)^2_{\text{MIN}} = E\{[I_{x_1z_1}{}^{(2)}]^2 + [I_{y_1z_1}{}^{(2)}]^2\} = (A_1 - C_1 - mz_0l)^2E\{\beta^2\} + (mz_0)^2\, E\{\alpha^2\} \quad (47)$$

Propellant Tank Misalignment

The rotor is balanced dry (without propellant) but with the propellant tanks symmetrically positioned about the bearing axis so that the addition of fuel does not disturb the balance. However, a slight position error in locating the tank centers will cause an imbalance to occur when the fuel is added. Assume there are three spherical tanks spaced 120° apart in a horizontal plane a distance l_p below the system c.g. and a distance r_p from the bearing axis. The cross products of inertia due to tank position errors Δx_i, Δy_i, Δz_i ($i = 1, 3$) take the form

$$I_{x_1z_1}{}^{(3)} = -\frac{m_p}{3}\sum_{i=1}^{3}(x_i\Delta z_i + z_i\Delta x_i)$$
$$I_{y_1z_1}{}^{(3)} = -\frac{m_p}{3}\sum_{i=1}^{3}(y_i\Delta z_i + z_i\Delta y_i)$$

where z_i is the distance of the ith tank from the plane of the system c.g. and x_i and y_i are its location in the horizontal plane with respect to the bearing axis. Assuming Δx_i, Δy_i, and Δz_i ($i = 1, 3$) are all independent zero mean variables with variance

$$E\{\Delta x_i^2 + \Delta y_i^2\} = E(\Delta r_p^2); \qquad E(\Delta z_i^2) = E(\Delta z_p^2)$$

the aforementioned cross product has statistics

$$[\theta^{(3)}]^2 \cdot (A_0 - C_1)^2_{\text{MIN}} = E\{[I_{x_1z_1}{}^{(3)}]^2 + [I_{y_1z_1}{}^{(3)}]^2\} = (m_p^2/3)\{r_p^2 \cdot E(\Delta z_p^2) + l_p^2 \cdot E(\Delta r_p^2)\} \quad (48)$$

Ballast Deployment Uncertainty

To achieve a desired inertia ratio, ballast must sometimes be deployed on extendable booms from the rotor following injection into orbit. The rotor is balanced in the laboratory with the ballast deployed; however, it is subsequently stored in-board for the launch. For a variety of reasons (e.g., thermal distortion, g droop, etc.), the ballast position following orbital deployment will deviate slightly from its position during balancing. Hence, a rotor unbalance will result which will contribute to the wobble magnitude. To evaluate this affect, assume ballast is deployed on three booms spaced 120° apart in a horizontal plane a distance l_b below the system c.g. and at a radial distance r_b from the bearing axis. Since the geometry is the same as in the propellant tank misalignment case, it follows immediately that

$$[\theta^{(4)}]^2 \cdot (A_0 - C_1)^2_{\text{MIN}} = (m_b^2/3) \cdot [r_b^2E(\Delta z_b^2) + l_b^2E(\Delta r_b^2)] \quad (49)$$

where $E(\Delta r_b^2)$ and $E(\Delta z_b^2)$ are the radial and axial variances in the deployment errors.

Bearing Runout

A final factor contributing to wobble is bearing runout. Because of bearing imperfections, the bearing shaft center line will not coincide with the housing center line and this lack of coincidence will produce a coning of the bearing axis. In this case, however, the coning angle is not necessarily constant, nor is its frequency necessarily equal to the rotor spin frequency. Rather the amplitude and frequency of the coning motion are functions of the various factors contributing to the shaft-housing center line misalignment, which factors are too numerous for any detailed treatment here. The usual procedure is to use the manufacturing tolerances to generate an upper bound on the applitude of the coning motion and then to combine this bound in root-sum-square fashion with the other wobble contribution. This procedure will be followed here even though it is not exactly correct. Thus, taking 0.005° as a typical upper bound for this effect provides the final wobble component as

$$\theta^{(5)} = 0.005° \quad (50)$$

The individual wobble components of Eqs. (46–50) are evaluated using the vehicle parameters in Table 1 and the 1σ values for the error sources as given in Table 2. The results are presented in Table 2 with the expected value of the wobble amplitude taking a value of 0.106°. Note that this amplitude is indeed small as was assumed in the development of Eqs. (32–35).

Conclusions

As has been shown, an out-of-balance rotor will cause the bearing axis of a dual spin vehicle to wobble about the system

Table 1 Vehicle mass properties

Symbol	Description	Value
m_1	rotor mass	1,300 lb
m_2	platform mass	500 lb
m	reduced mass: $m_1m_2/m_1 + m_2$	361.1 lb
m_p	propellant mass	25 lb
m_b	ballast mass	250 lb
A	vehicle transverse inertia	236 slug-ft²
A_1	rotor transverse inertia	140 slug-ft²
C_1	rotor axial inertia	260 slug-ft²
z_0	z component of $\mathbf{r}_0$ [Eq. (29)]	−27 in.
l	distance from alignment plane to rotor c.g.	16 in.
l_p	distance from c.g. plane to fuel plane	17 in.
r_p	radial distance of fuel tanks from bearing axis	20 in.
l_b	distance from c.g. plane to ballast plane	24 in.
r_b	radial distance of ballast from bearing axis	60 in.

Table 2 Summary of wobble contributors

Wobble error source	Accuracy requirement (1σ value)	Wobble angle, deg
1) Balance machine residuals	Residual unbalance = 14 lb-in.^{2} Residual c.g. offset = 0.0005 in.	0.0077
2) Bearing axis tilt and offset	Tilt = .01° Offset = .002 in.	0.037
3) Propellant tank misalignment	In-plane = 0.05 in. Out-of-plane = 0.05 in.	0.0098
4) Ballast deployment uncertainty	In-plane = 0.05 in. Out-of-plane = 0.01 in.	0.099
5) Bearing runout	...	0.005
	Total RSS value	0.106

angular momentum vector. If the vehicle has equal transverse inertias about the x_2y_2 platform coordinate axes, the wobble angle is a constant in magnitude and varies in direction at rotor spin speed. If the transverse inertias differ, then the wobble motion is elliptical; that is, the bearing axis locus is a right cone but with elliptical cross section.

The unbalance which produces the motion is due to two effects: a misalignment of the bearing and rotor principal axes (dynamic unbalance), and an offset of the rotor c.g. from the bearing axis (static unbalance). As indicated in the example problem, a variety of factors arise during the balancing process which contribute to the rotor unbalance. However, only minimal knowledge regarding the statistics of each contributing factor is required to develop an estimate of the wobble magnitude. If this estimate exceeds some specified limit for the mission under consideration, then either the balancing process and equipment must be improved or an active subsystem must be included for in-orbit balancing.

In developing the wobble magnitude it has been assumed that the rotor is axisymmetric about its own principal axis; that is, that the rotor principal inertia matrix takes the form

$$(I_1)_{\mathrm{PRIN}} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix}$$

In almost all applications the two transverse inertias will differ slightly with $(I_1)_{\mathrm{PRIN}}$ becoming

$$(I_1)_{\mathrm{PRIN}} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix}$$

If the difference between A_1 and B_1 is small, then the analysis of the preceding sections is valid (to first order) provided the parameter A_1 is replaced by $(A_1 + B_1)/2$. That is, the rotor transverse inertia appearing in the analysis should be the average of the transverse inertias in any two orthogonal directions.

References

[1] Iorillo, A. J., "Nutation Damping Dynamics of Rotor Stabilized Satellites," paper presented at the ASME Annual Winter Meeting, Chicago, Ill., Nov. 1965.

[2] Likins, P. W., "Attitude Stability Criteria for Dual Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 4, No. 12, Dec. 1967, pp. 1638–1643.

[3] Mingori, D. L., "Effects of Energy Dissipation on the Attitude Stability of Dual Spin Satellites," *AIAA Journal*, Vol. 7, No. 1, Jan. 1969, pp. 20–27.

[4] Mingori, D. L., Harrison, J. A., and Tseng, G. T., "Semipassive and Active Nutation Dampers for Dual Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 8, No. 5, May 1971, pp. 448–455.

[5] Likins, P. W., Tseng, G. T., and Mingori, D. L., "Stable Limit Cycles Due to Nonlinear Damping in Dual Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 8, No. 6, June 1971, pp. 568–574.

[6] Stadelbauer, D., "Mass Properties Determination on Aerospace Balancing Systems," *The Weight Record*, Vol. 26, No. 2, April 1967, pp. 23–40.

[7] Jones, D. L., "Dynamic Balance Error Terms for Rotating Aerospace Vehicles," paper 740 presented at the 28th Annual Conference of the Society of Aeronautical Weight Engineers, San Francisco, Calif., May 1969.

[8] Kelley, O. A., "Comments on Aerospace Vehicle Dynamics Balancing," paper 772 presented at the 28th Annual Conference of the Society of Aeronautical Weight Engineers, San Francisco, Calif., May 1969.